Control parameter in granular convection

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We perform dynamic simulations of discrete systems involving convection roll patterns. The present study reveals that the self-diffusion constant D and the energy diffusivity α are two transport coefficients relevant to dynamic granular systems. This conforms to the basic physics of vibrated granules: a phenomenon of mass diffusion induced by energy injection. The ratio D/α gives rise to an intrinsic physical property which, together with the external vibrational acceleration, constitutes a dimensionless control parameter equivalent to the Rayleigh number in heat induced fluid convection. [S1063-651X(98)05810-3]

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I. INTRODUCTION

Over the past decade there has been a surge of interest in the basic physics of pattern formation in diverse fields, including crystallography, geology, hydrodynamics, biology, and recently in granular materials under dynamic conditions. Rayleigh-Bénard convection rolls, which emerge when a thin layer of fluid is heated from below, have long been studied as a representative case of spatial pattern formation in nonequilibrium fluid systems [1]. The instability in fluid motion, which is the precursor to pattern formation, is governed by the Rayleigh number R, which is the ratio of the destabilizing buoyancy force to the stabilizing dissipative force.

Recently it has been found that multipairs of convection rolls appear when granules in a rectangular container are subjected to (continual) vertical vibrations [2]. Although granules are often fluidized in a variety of industrial processes, there exists no consensus yet as to intrinsic parameters, like viscosity in fluid, that characterize the dynamic state of granules. The present paper conducts two dimensional particle dynamics simulations to search for such a control parameter for granular materials under dynamic conditions.

The duality of granular behavior, both fluidlike and solidlike, has attracted the attention of a growing number of researchers [3–5]. Heap formation accompanied by convection currents, which occurs when granules are subjected to vertical vibration, is one well known example of granular behavior that shows the mixed characteristics, both fluidlike and solidlike. Another good example that shows the duality of granules is multipairs of convection rolls, analogous to Rayleigh-Bénard convection in fluids [2], which appear in a wide size range of granules when vibrated. There are, however, certain aspects in the granular convection that differ significantly from convection in fluids. Two particular characteristics of granular convection rolls are (i) the number of convection rolls increases with increasing vibrational intensity with other conditions fixed, and (ii) there are, in certain cases, large dead zones in between convection rolls or near the side walls, where granules hardly move. Several attempts have been made to express dynamic granular behavior by continuum theory using modified Navier-Stokes equations [6,7]. However, as argued by Jaeger *et al.* [5], there is no compelling reason to believe that the granular system can be characterized by a continuum model. In the granular systems under consideration energy is solely supplied by external vibrations since the thermal fluctuations of the constituent particles are negligible. The local fluctuation and macroscopic flow of particles are both induced by the external vibrational force acting on the system. In contrast, heat (thermal fluctuation) and work (movement of the control volume) are distinguishable in fluid dynamics. The major problem in describing the granular system by a continuum model lies in the difficulty of differentiating the microscopic length scale from the macroscopic one: the ratio of characteristic length of the pattern to particle size extends only to the order of 10^3. Under such a condition it becomes difficult to make clear-cut definitions for intrinsic physical properties of the system, such as viscosity.

In addition, the difficulty of describing the behaviors of granules in the context of existing physical laws is further enhanced by the following reasons. A ''dynamic'' reference state, such as the thermal equilibrium in thermodynamic systems, is not known in granular systems. The static state is the only relevant reference state, although even multiple states exist as static states depending on how much stress is accumulated in the granules. Furthermore, the static physical properties are not necessarily related to the physical properties of dynamic systems; it is not certain what kinds of variables change continuously (or slowly) in granular systems. For certain granular systems there exists a dynamic steady state where the energy input and the dissipation balance. The convective roll pattern in a vertically vibrated bed, which we treat in this paper, is one such example. However, even in such a case, it is not clear whether the average can be used to describe the system effectively. An ensemble average at a given instant does not necessarily reflect the entire dynamical process of the system [8] and thus the ensemble average seems to be insufficient to specify the dynamic granular state. If there exists a dynamic reference state, it will probably be defined as a time average of the system.

The critical point of instability in open systems, at which

an ordered pattern emerges, is determined by the balance between the destabilizing force and the stabilizing force. The stabilizing force is dissipative in nature, and is associated with intrinsic physical properties of the system. In the case of a vibrated granular bed the destabilizing force is obviously the vibrational one. A widely used external dimensionless control parameter for vibrated granular beds is the ratio of vibrational acceleration to that due to gravity $\Gamma = a(2\pi f)^2/g$, where a and f are the amplitude and frequency of the vibration, respectively, and g is the gravitational acceleration. As to the intrinsic physical property, however, there is no universally accepted one, like viscosity in fluid, that can be readily measured. Since the granules (because of its discreteness) dilate and/or are thrown into space with the movement of the container, the transfer rate of vibrational energy to the bed varies with time, and hence static properties are not directly related to the dynamic behavior of granules. Therefore certain parameters, which reflect the effectiveness of energy transfer to the bed along with diffusivity of the particles, need to be specified. The objective of the present study is to evaluate such parameters, and to search for a universal control parameter characterizing the vibrated granular bed. Underlying the present study is the question of whether we can define a macroscopic variable (expressing the physical properties of granules) independently of microscopic details (despite the fact that the granular system exhibits a mixed state, both fluidlike and solidlike characteristics).

II. METHOD

A. Simulation model

Particle dynamics (PD) simulation uses a model which defines the interaction between particles (usually in a form of pairwise interaction). The dynamics of the system is determined by the force (which is the sum of all pairwise interactions) acting on each particle. Simulating a system consisting of many particles which are interacting via a simple model as described above has the advantage of extracting the essential features of the phenomena under consideration. We use a model which is based on the postulates that basic features of granular materials can be described by the excluded volume effect and the dissipation of kinetic energy between granules [9–11]. The excluded volume effect can be defined as a property that the force f_{ij} between two particles i and j tends to be infinity as r_{ij} approaches a limiting value, where r_{ij} is a distance between particles i and j. Both hard-core (which is $f_{ij}=\infty$ at $r_{ij}=d$) and soft-core ($f_{ij}=\infty$ at $r_{ij}=0$) models meet this criterion. Such simple models express a sort of degenerate phase space whose parameters are fewer than those of the experimental real space. (For instance, hard-core models do not have an independent unit of time; it must be determined in relation to kinetic energy.) Therefore we encounter the difficulty of matching the intrinsic simulation units with the real space units. Indeed, this matching is sometimes crucial whether or not the simulation succeeds in accurately reproducing the experimentally observed phenomena. We have succeeded in establishing the relation between intrinsic simulation units and experimental ones in our previous study through the frequency dependence of surface wave patterns [10]. It should be mentioned that the dissipation coefficient γ (which is the only internal parameter in our model) expresses the difference in material property. Previous studies have shown that the effect of granular size can be accounted for by the difference in the value of dissipation coefficients. This implies that the simulation with a change in the value of γ represents either (i) a system consisting of particles made of different material (from the original) with the particle size fixed or (ii) a system of different sized particles (from the original) made of the same material.

We perform simulations on dynamic systems corresponding to glass bead beds under vibrations at frequency 50 Hz [$g_s = 0.2352/(2\pi)^2$, refer to [10]]. This is to compare simulation results with experiments [2]. The use of an algorithm suited for vectorization [12] enables us to pursue a systematic investigation of large systems by workstations. Here we report on simulation results of the following systems: (i) dissipation coefficient $\gamma=0.4$, system size $N=9000$, and container width $L=150$; (ii) $\gamma=0.4$, $N=12\,000$, and $L=200$; (iii) $\gamma=1.0$, $N=9000$, and $L=150$; (iv) $\gamma=2.0$, $N=12\,000$, and $L=200$; (v) $\gamma=0.4$, $N=4500$, and $L=150$; (vi) $\gamma=0.4$, $N=13\,500$, and $L=150$; (vii) $\gamma=0.4$, $N=18\,000$, and $L=150$. As mentioned above, the difference in the value of γ expresses the difference in material property and/or granular size. Results are shown in reduced units. As a representative case, velocity profiles of system (i) under various vibrating intensities are shown in Fig. 1. Figure 1 indicates that our simulations reproduce well basic features of convection rolls observed in the experiment [2]. Note that the number of rolls increases with increasing vibrational amplitude. This is the first to show the multiple pairs of convection rolls via computer simulations.

B. Transport coefficients

Here we investigate whether the first order transport coefficients can be extracted by particle dynamics simulation of vibrated granules.

In general, transport coefficients can be described, to a first order, by the phenomenological relationship

$$\frac{\partial X}{\partial t} = c\frac{\partial^2 X}{\partial x^2}, \tag{1}$$

where X stands for concentration for mass transfer, and energy for energy transfer, with the coefficient c being mass diffusivity (constant of self-diffusion) in the former, and energy diffusivity (related to thermal diffusivity in fluids [13]) in the latter.

Expressions for the transport coefficients involving dissipation in canonical ensemble were given by Helfand [14]. The transport coefficients were also investigated by Alder *et al.* [15] using molecular dynamics simulations. The above studies use the finding of statistical physics of thermodynamic equilibrium, i.e., $\langle \Delta E^2 \rangle = kT^2 C_v$. In an open system operating far from thermodynamic equilibrium such a relationship cannot be utilized. However, when one deals with transport phenomena involving a sufficiently long time scale compared to the characteristic time of perturbation (in this study, vibrational period) the phenomenological transport coefficients can be evaluated by the slopes determined by the relations given below. The mass diffusivity D is related to

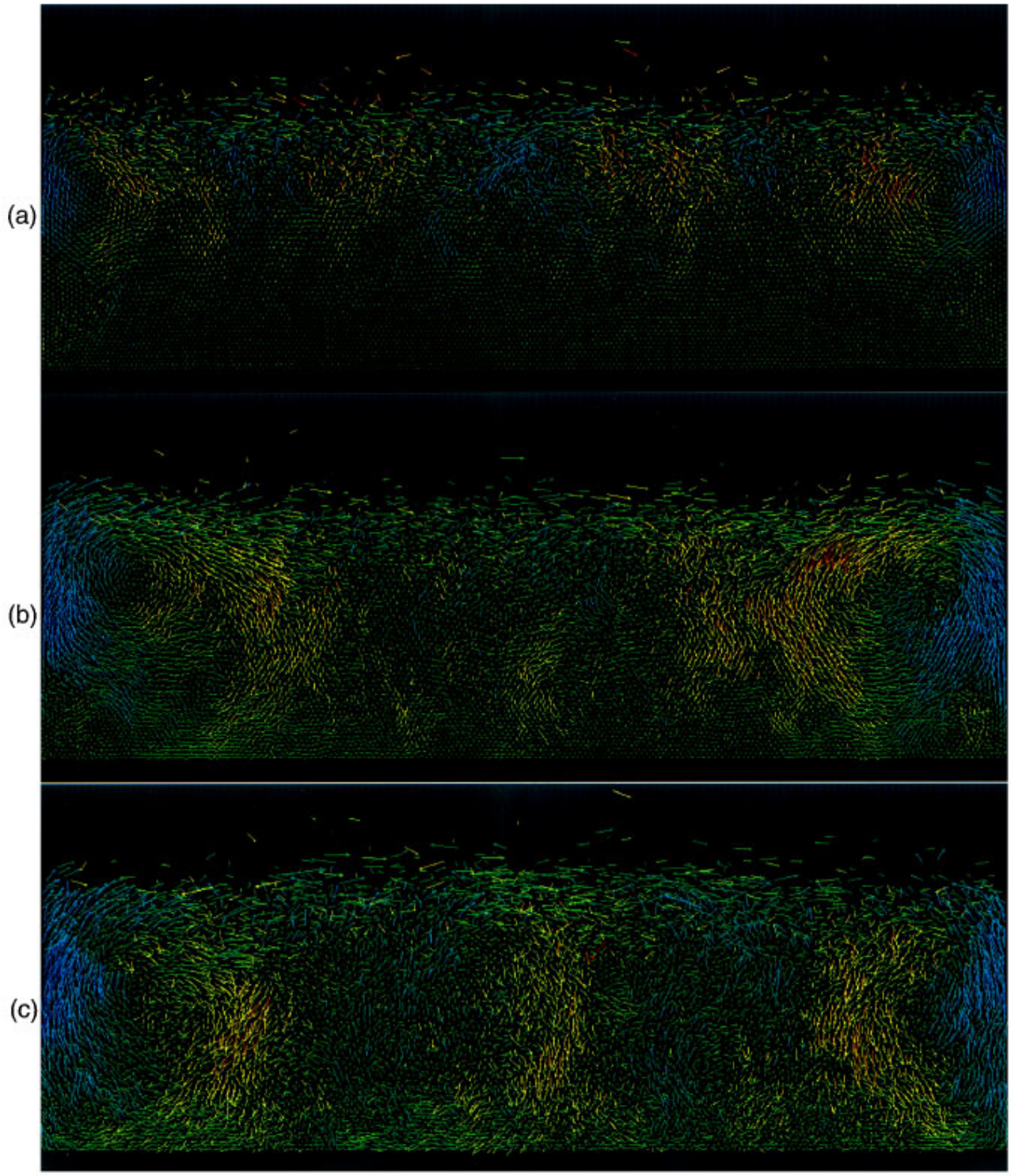

FIG. 1. (Color) Velocity profiles of vertically vibrated granular beds of system (i) $\gamma=0.4$, $N=9000$, and $L=150$ for three values of Γ. Colors show relative magnitude of vertical velocities: blue denotes the maximum upward flow velocity and red the downward. The average velocity vectors for 140 vibration cycles are shown. (a) $\Gamma=3$: Only the upper half of the bed fluidizes and particles near the bottom do not diffuse. $\Gamma=3$ is slightly lower than the threshold value for the appearance of stable convective roll patterns. (b) $\Gamma=4$: Two convection rolls can be discerned near the walls. There exists a dead zone where particles hardly diffuse in the middle. (c) $\Gamma=6$: Highly fluidized state with six rolls (four upward and three downward flows).

the mean square displacement as shown below.

$$D=\langle[x(t)-x(0)]^2\rangle/2t, \tag{2}$$

where $x(t)$ denotes the position at time t, and the angular brackets indicate an average over an ensemble: an average over the entire system at a given time t.

Following a procedure similar to Helfand [14], we obtain

the expression for energy diffusivity below. Here, ''ensemble'' is used to mean a collection of states; it is not necessarily the true microscopic ensemble in the sense of statistical mechanics. This paper tries to follow the phenomenological approach as much as possible, since there is hardly any established theory concerning the statistical mechanics on fluidized granules. We start from the phenomenological relation of first order,

$$\frac{\partial E}{\partial t}=\alpha_x\frac{\partial^2 E}{\partial x^2}, \tag{3}$$

where $E(t)$ is the internal energy of a particle at time t. We consider a subensemble whose initial position is x_0 and energy E_0, and is subject to boundary conditions

$$E(x,0)=E_0\delta(x-x_0), \quad E(\pm\infty,t)=0. \tag{4}$$

The second boundary condition in Eq. (4) indicates that the subensemble (x_0,E_0) cannot diffuse to infinity in finite time. The solution for Eq. (3) subjected to the above boundary conditions is

$$E=\frac{E_0}{\sqrt{4\pi\alpha_x t}}\exp\left[-\frac{(x-x_0)^2}{4\alpha_x t}\right], \tag{5}$$

which shows that the broadening of the original δ function is Gaussian. Equation (5) also indicates that the energy of subensemble (x_0,E_0) decays to zero as $t\to\infty$. In continuously vibrated granular systems the consecutive perturbation (energy input) might occur at a finite time t (before the input energy decays to zero). Previous simulation studies show that the time instant when the energy input occurs will be position dependent: it is especially dependent on the vertical position of particles in the bed [9].

The second moment of E is

$$M_2=\int_{-\infty}^{\infty}(x-x_0)^2E\,dx$$
$$=\int_{-\infty}^{\infty}\frac{E_0}{\sqrt{4\pi\alpha_x t}}(x-x_0)^2\exp\left[-\frac{(x-x_0)^2}{4\alpha_x t}\right]dx=2E_0\alpha_x t. \tag{6}$$

Reassembling the ensemble, using the above result [Eq. (6)] and the relationship $f(x_0)=1/l_x$, we have

$$I=\sum_i^N\int_0^{l_x}\int_{-\infty}^{\infty}E_0M_2f(x_0)f(E_0)dx_0dE_0$$
$$=2\alpha_x t\sum_i^N\int_{-\infty}^{\infty}E_0^2f(E_0)dE_0=2\alpha_x t\langle E^2\rangle. \tag{7}$$

No assumption is made as to the probability of the subensemble (x_0,E_0), $f(x_0)f(E_0)$. Microscopically, I can be interpreted as

$$I=\langle E_i(0)[x_j(t)-x_i(0)]^2E_j(t)\rangle. \tag{8}$$

Combining Eqs. (7) and (8) leads to

$$\alpha_x=\frac{\langle[x_j(t)-x_i(0)]^2E_i(0)E_j(t)\rangle}{2\langle E^2\rangle t}, \tag{9}$$

where α_x is the energy diffusivity in horizontal (x) direction [16]. Measurements of diffusivities in our PD simulations are done under the assumption that the correlation of particles of $i\neq j$ is not relevant in long time scales. The value $\Sigma_i[x_i(t)-x_i(0)]^2E_i(0)E_i(t)/N$ (summation over i alone) is measured in our simulation in relation to the energy diffusivity. If the phenomenological description of Eq. (3) holds in granular systems, the value $\langle[x(t)-x(0)]^2E(0)E(t)\rangle$ should increase linearly with t in a dynamic steady state. For simplicity only the horizontal (x) components of the D and α are given above. In the simulations, both the horizontal (x) and vertical (y) components were evaluated to estimate the degree of anisotropy of the system. The simulated values of mean square displacements and $\langle[x(t)-x(0)]^2E(0)E(t)\rangle$ fluctuate, whose magnitudes generally increase with time except for a few cases (explained later). In spite of such fluctuations, their mean values increase almost linearly with time. Therefore α_x in Eq. (9) can be determined by taking the straight-line fit (in terms of mean square) from $\langle[x(t)-x(0)]^2E(0)E(t)\rangle$ versus t plotting when the system is in a dynamic steady state [17]. When taking the ensemble average of the energy $\langle E(t)\rangle$ over time t, we observe that it fluctuates in an unpredictable manner in a short time span. As a physical property representing the dynamic steady state of the system the long time average of the entire system is considered to be more appropriate than the ensemble average at a given time instant $\langle E(t)\rangle$. Thus the value $\int_0^s\langle E^2(t)\rangle dt/s$ for a time period s is used instead of the value $\langle E^2(t)\rangle$ in Eq. (9): s is taken to be the time period in which physical properties are measured (usually 300 vibrational periods).

It is not appropriate to interpret α as a parameter related to thermophysical variables, since the thermal fluctuation (heat) of the constituent particles is negligible in granular systems.

III. RESULTS

A. Internal energy

First, we show that the commonly used parameter Γ is not a universal parameter in that Γ alone cannot specify a state (degree of fluidization) of the bed independently of granular materials. The relations between Γ and an ensemble average of internal energy $\langle E\rangle$ and its variance $\sigma=\sqrt{\langle E^2\rangle-\langle E\rangle^2}$ (over 300 vibrational cycles) at dynamic steady states are shown in Figs. 2(a) and 2(b), respectively; the internal energy E is defined as the kinetic energy plus the potential energy. Figure 2 indicates that a linear relationship holds between $\langle E\rangle$ and Γ (and between σ and Γ) for each system, but its proportional constant differs between systems with different γ. This implies that the dynamic state of the bed (represented by $\langle E\rangle$) is dependent on the granular material. It is evident on inspection of Fig. 2 that the differences in the values of $\langle E\rangle$ become insignificant for small values of Γ, indicating that the dynamic state of the bed does not depend on material for small values of Γ. It is reported that the critical value of Γ for heap formation, which is slightly above unity, is independent of particle size [18], which, in

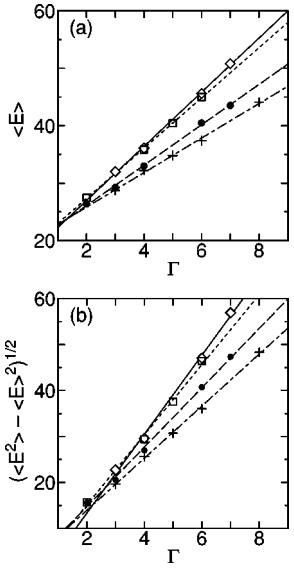

FIG. 2. Ensemble average of the internal energy E (a) and its variance (b) over 300 cycles of vibration versus Γ for systems (i) $\gamma=0.4$, $N=9000$, and $L=150$ ($\diamond$, solid line); (ii) $\gamma=0.4$, $N=12\,000$, and $L=200$ ($\square$, dotted line); (iii) $\gamma=1.0$, $N=9000$, and $L=150$ ($\bullet$, dashed line); (iv) $\gamma=2.0$, $N=12\,000$, and $L=200$ ($+$, dot-dash line). Lines represent least mean square fits.

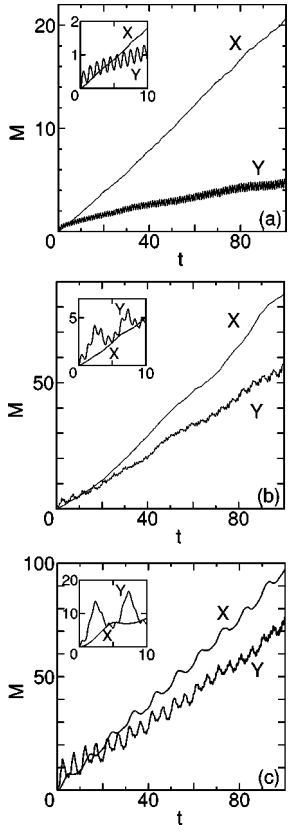

FIG. 3. Mean square displacements (M's) versus time t (in units of a vibration cycle) in horizontal (X) and vertical (Y) directions for system (ii) at (a) $\Gamma=2$ (amplitude $a=0.47$), (b) $\Gamma=4$ ($a=0.94$), and (c) $\Gamma=6$ ($a=1.41$).

fact, conforms to our simulation results in that the dynamic state of the system (in our simulations) is scarcely dependent on γ (note that the difference in the value of γ means difference in either material or particle size) for $\Gamma\leqslant 2$, but is clearly dependent on γ for larger values of Γ.

It should also be mentioned that the critical value of Γ, above which the convection roll patterns appear, is dependent on granular materials (the value of γ), in good agreement with experiment.

The above results clearly show that the commonly used parameter Γ is not a universal parameter: Γ alone cannot specify a state of the bed independently of granular materials.

B. Transport properties

In this section we describe the time evolution of the physical values pertinent to the transport properties of the dynamic state of granular systems, and how we evaluate the first order phenomenological transport coefficients.

In Figs. 3(a), 3(b), and 3(c), we show mean square displacements $M=\langle[x(t)-x(0)]^2\rangle$ as a function of time t in both horizontal (denoted by X) and vertical (denoted by Y) directions for system (ii) at $\Gamma=2$, 4, and 6, respectively. Time t is expressed in a unit of one vibrational cycle. This time unit will be used hereafter unless stated otherwise. Each line in Fig. 3 represents an average of three consecutive runs, each with 100 vibrational cycles after the system has reached a dynamic steady state. Insets show the system evolution at early times $t\leqslant 10$. At $\Gamma=2$, the particles move only near the free surface and container walls, and the resultant diffusion constants (which were evaluated from the slope of M's) exhibit strong anisotropy $D_x/D_y=5.0$ [Fig. 3(a)]. Note that the diffusion in horizontal (X) direction is smooth, while the value of M in vertical (Y) direction is resonant with the external vibration. However, the mean values of M's increase almost linearly with time, allowing us to determine the diffusion constants D_x and D_y from their angles of inclination to the abscissa. At $\Gamma=4$ [Fig. 3(b)], the diffusion in X direction is still smooth (although the linearity with respect to t deteriorates compared to that when $\Gamma=2$), but the values of M in Y direction fluctuate with a period five times the external vibration as evidenced in the inset. Convective roll patterns appear for this condition of granular beds, although the roll patterns are not so stable. The fluctuation over a long time (five times the vibration period) of M in vertical (Y) direction becomes more clearly noticeable at $\Gamma=6$ as seen in Fig. 3(c). It is also seen in Fig. 3(c) that a long time fluctuation (ten times the vibration period) appears for the horizontal (X) direction as well. Six convection rolls are seen to appear for the case of Fig. 3(c).

The linearity of the M's with t is dependent on the vibrational acceleration Γ and material property γ. There is a tendency that the state of the granular bed becomes unstable with increasing values of Γ and γ, resulting in greater irregularities in the values of M's. Nevertheless, we use the angle of inclination in the plotting of M vs time to evaluate phenomenological (first order) diffusion constants.

We examine the relationship, Eq. (9), in this paragraph. Simulation results of $\langle[r(t)-r(0)]^2E(t)E(0)\rangle$ are shown in Fig. 4 as a function of t for system (ii) at (a) $\Gamma=2$ in both horizontal (denoted by X) and vertical (denoted by Y) directions, (b) $\Gamma=4$ in horizontal direction, (c) $\Gamma=4$ in vertical direction, (d) $\Gamma=6$ in horizontal direction, and (e) $\Gamma=6$ in vertical direction, respectively. All data shown are averages

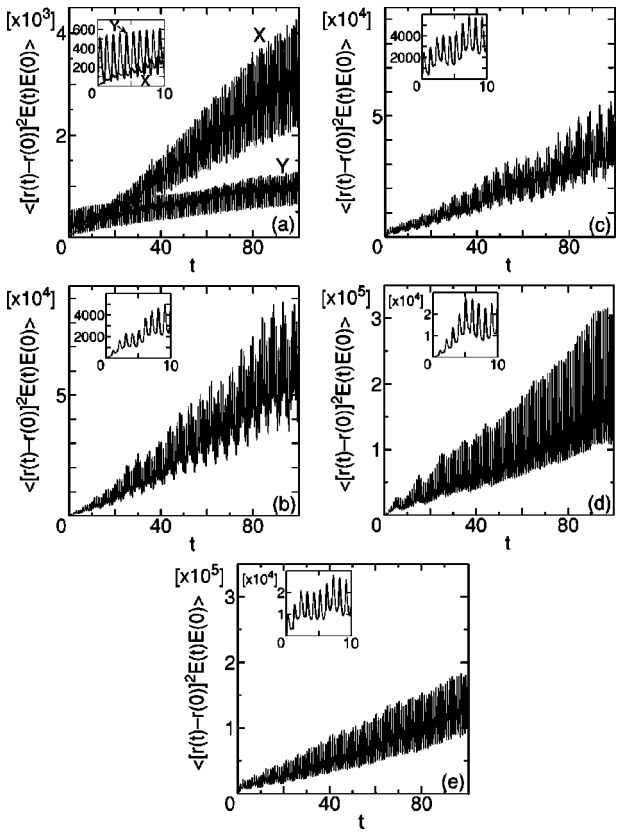

FIG. 4. $\langle[x(t)-x(0)]^2E(0)E(t)\rangle$ versus time t for system (ii) at (a) $\Gamma=2$ (amplitude $a=0.47$) in horizontal (X) and vertical (Y) directions, (b) $\Gamma=4$ ($a=0.94$) in horizontal direction, (c) $\Gamma=4$ in vertical direction, (d) $\Gamma=6$ ($a=1.41$) in horizontal direction, and (e) $\Gamma=6$ in vertical direction.

of three consecutive runs, each with 100 vibrational cycles after the system has reached a dynamic steady state. Insets show the time evolution at early times $t\leqslant 10$. The value $\langle[r(t)-r(0)]^2E(t)E(0)\rangle$ shows a resonance with the external vibration whose phase in the horizontal direction differs by approximately π from that in the vertical direction. Fluctuations of these values as well as their magnitudes generally increase with t except for the case of $\Gamma=2$ in vertical direction, when the fluctuations remain at nearly the same level.

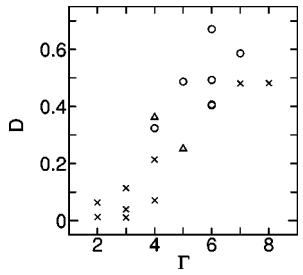

FIG. 5. Self-diffusion constant D versus Γ. Different symbols refer to different states of the system; ○: stable convection rolls appear; △: convection rolls appear but are not stable; ×: no stable convection roll (either partially fluidized or highly agitated) appears. Results for systems (i) – (iv) are shown. The values of D and α were evaluated by taking averages of three consecutive runs, each with 100 vibrational cycles after the system has reached a dynamic steady state.

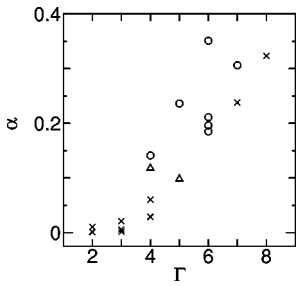

FIG. 6. Energy diffusivity constant α versus Γ. Symbols ○, △, and × refer to the same states of the system explained in Fig. 5. Results for systems (i) – (iv) are shown.

The value of α is calculated from the mean angle of the inclination in the plotting of $\langle[r(t)-r(0)]^2E(t)E(0)\rangle$ vs time.

The self-diffusion constant D and the energy diffusivity α for systems (i)–(iv) are plotted against Γ in Figs. 5 and 6, respectively, wherein symbols ○, △, and × are used to express the states of convection for all the systems investigated. Figures 5 and 6 both show that the values of D and α are generally larger (at given values of Γ) when stable convection rolls exist than when no stable convection roll exists. However, no clear-cut relation is discernible between the diffusivities and Γ. We have presented in [19] how the material property affects D (and α) vs Γ relations.

Since granular convection can be envisaged as particle mass diffusion induced by energy input, the ratio of D and α (both of which have the dimension of length^2/time) can be thought of as a relevant dimensionless parameter for the system. The values of D/α are plotted against Γ in Fig. 7 [data for systems (i) – (iv) are included in Fig. 7]. It can be seen from Fig. 7 that the value of D/α is larger than 5 when $\Gamma\leqslant 3$ but becomes significantly small for larger values of Γ. This large difference in the values of D/α for $\Gamma\leqslant 3$ and for $\Gamma\geqslant 4$ suggests that D/α manifests the intrinsic physical property of the dynamic granular systems, because granular systems exhibit rather strong resistance (whose characteristics depend on material properties) against perturbations (due to external forces) at the quasistatic regime $\Gamma\leqslant 3$: this strong resistance is well represented by significantly large values of D/α when $\Gamma\leqslant 3$. An important point to note is that the average value of D/α (for the entire system in 300 vibrational cycles) characterizes a macroscopic physical property of the system well, although each of its constituents D and α differs locally (or microscopically). The value of D/α charac-

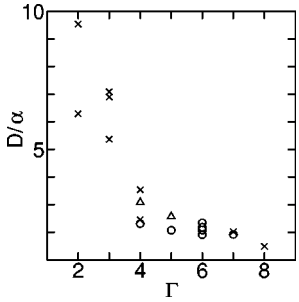

FIG. 7. D/α versus Γ. Symbols ○, △, and × refer to the same states of the system explained in Fig. 5. Results for systems (i) – (iv) are shown.

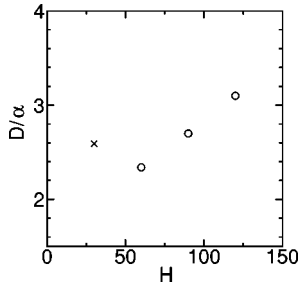

FIG. 8. D/α versus H at $\Gamma=6$ for $\gamma=0.4$ and $L=150$. Symbols ○, △, and × refer to the same states of the system explained in Fig. 5. Results for systems (i), (v), (vi), and (vii) are shown.

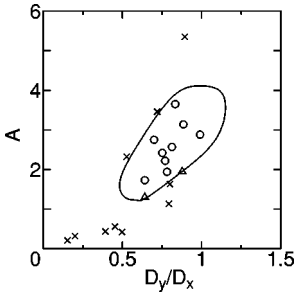

FIG. 9. The dynamical anisotropy D_y/D_x versus $\mathcal{A}=\Gamma\alpha/D$. Symbols ○, △, and × refer to the same states of the system explained in Fig. 5. Results for systems (i) – (vii) are shown. Stable convection roll patterns appear within the closed line.

terizes the internal physical property of the dynamic granular systems, whereas viscosity characterizes the physical property of fluids in motion: D/α in granular dynamics and viscosity in fluids play a similar role in the sense that both characterize physical properties of pertinent systems macroscopically (independently of microscopic details). We have shown in [19] how the material property (the value of γ) affects the relation between D/α and Γ. The material property strongly affected D/α especially in the quasistatic regime ($\Gamma\leqslant 3$).

Experimental studies have shown that the bed height H of the granular bed is an important factor that characterizes the state diagram of a granular system [2]. In Figs. 5–7 we have dealt with systems different in material but identical in the bed height $H=N/L$. Figure 8 shows D/α as a function of H for a given material ($\gamma=0.4$) at $\Gamma=6$. We observed six, four, and two stable convection rolls appear for $H=60$, 90, and 120, respectively. Figure 8 clearly shows that D/α is dependent on H. This is probably because the density of the granular assemblage increases with increasing bed height, thereby enhancing the collective motion of granules. Comparison between Figs. 7 and 8, at $\Gamma=6$ (which represents a highly fluidized regime), indicates that difference in the bed height H affects D/α more drastically than does difference in material property. This contrasts with the behavior in the quasistatic regime ($\Gamma\leqslant 3$) where the difference in material property has drastic effects on the value of D/α.

TABLE I. Self-diffusion constant D and energy diffusivity α in horizontal (x) and vertical (y) directions and the control parameter $\mathcal{A}=\Gamma\overline{\alpha}/\overline{D}$.

System	Γ	$\langle E\rangle$ [$\times 10^1$]	$\langle E^2\rangle$ [$\times 10^3$]	D_x [$\times 10^{-1}$]	D_y [$\times 10^{-1}$]	α_x [$\times 10^{-1}$]	α_y [$\times 10^{-1}$]	D_x/α_x	D_y/α_y	$\mathcal{A}$	State
	3	3.20	1.54	1.57	0.709	0.273	0.151	5.75	4.68	0.559	×
(i)	4	3.61	2.18	3.95	2.53	1.80	1.01	2.19	2.51	1.73	○
	6	4.56	4.29	5.44	4.42	2.44	1.78	2.23	2.48	2.57	○
	7	5.08	5.82	5.60	4.03	2.76	2.00	2.03	2.01	3.46	×
	2	2.75	1.00	1.06	0.211	0.165	0.0364	6.40	5.79	0.318	×
(ii)	4	3.58	2.15	4.41	2.83	1.48	0.874	2.99	3.24	1.30	△
	5	4.04	3.05	5.57	4.17	2.79	1.93	2.00	2.17	2.42	○
	6	4.50	4.18	4.75	3.32	2.19	1.51	2.17	2.20	2.75	○
	2	2.64	0.941	0.218	0.0332	0.0213	0.00499	10.2	6.66	0.21	×
	3	2.92	1.28	0.567	0.222	0.0739	0.0405	7.67	5.50	0.435	×
(iii)	4	3.30	1.82	2.38	1.90	0.648	0.559	3.68	3.39	1.13	×
	6	4.05	3.30	7.11	6.30	3.90	3.13	1.82	2.01	3.14	○
	7	4.35	4.14	6.41	5.32	4.17	1.96	1.54	2.72	3.65	○
	3	2.87	1.21	0.144	0.0719	0.0184	0.0121	7.85	5.93	0.42	×
	4	3.22	1.70	0.785	0.627	0.333	0.243	2.36	2.58	1.63	×
(iv)	5	3.48	2.15	2.69	2.35	1.08	0.881	2.49	2.67	1.95	△
	6	3.74	2.70	4.09	4.05	2.09	1.83	1.96	2.22	2.88	○
	8	4.41	4.28	5.10	4.55	3.75	2.70	1.36	1.68	5.35	×
(v)	6	3.65	3.18	9.69	5.11	3.49	2.22	2.78	2.30	2.32	×
(vi)	6	5.51	5.70	12.1	9.33	4.96	2.95	2.44	3.16	2.22	○
(vii)	6	6.42	7.28	21.6	16.9	7.42	4.97	2.91	3.39	1.94	○

C. Control parameter

From the results mentioned above, it is clear that neither the intensity of external vibration Γ, nor the internal parameter D/α alone can express the degree of fluidization, since the bed shows different degrees of fluidization for given values of Γ or D/α. Naturally, the control parameter that determines the critical point of instability is dependent on both the external and the internal parameters that constitute a given dynamic system, thus we introduce a dimensionless control parameter for the granular convection:

$$\mathcal{A}=\frac{\Gamma}{D/\alpha}=\frac{a(2\pi f)^2\alpha}{gD}. \tag{10}$$

The dimensionless parameter $\mathcal{A}$ is the ratio of the destabilizing vibrational force $Ma(2\pi f)^2$ to the stabilizing force MgD/α, where M denotes the effective bed mass.

The Rayleigh number in a continuum fluid depends on the temperature variation of the fluid over a given length scale. Similarly, in fluidized granules, the control parameter $\mathcal{A}$ depends on Γ and the bed height H for a given granular material. However, H does not explicitly appear in the control parameter $\mathcal{A}$: it is implicitly included in $\mathcal{A}$ through D/α as evidenced in Fig. 8.

We now show the stability diagram of granular convection in Fig. 9, wherein the dynamic anisotropy D_y/D_x corresponds to the wave vector in the convection roll patterns in fluids: the wave vector (anisotropy of the rolls) cannot be estimated from the number of convection rolls and the aspect ratio of the container when dead zones exist. It can be seen from Fig. 9 that there exists a region, analogous to the Busse balloon in heat induced convection, where the stable roll convection persists. The critical value of $\mathcal{A}$ is around 1.3.

IV. CONCLUDING REMARKS

We have shown that the state of granular convection involving multipairs of convection rolls is governed by a control parameter $\mathcal{A}$. The ratio of two nondimensional parameters (the intensity of external vibration to the internal physical property) relevant to granular dynamics constitutes the control parameter $\mathcal{A}$. The parameter $\mathcal{A}$ can also be interpreted as the ratio of the destabilizing vibrational force to the stabilizing force. The physical meaning of $\mathcal{A}$ is thus similar to the Rayleigh number of heat induced convection. However, the macroscopic variables characterizing the internal physical properties differ between the two cases. In granular dynamics, thermal fluctuation of the constituent particles is negligible, and the local fluctuation and macroscopic flow of granules are both induced by external vibrational force. Thus in contrast to fluid systems, thermophysical variables are irrelevant in the physics of granular materials. D/α is found to be a representative variable that characterizes the internal physical properties for vibrated granular beds. The intrinsic physical property of the dynamic state, represented by D/α, is not only dependent on the microscopic property of the granules but also dependent on the density of the granular assemblage which, in turn, is dependent on Γ and H. We hope that this study will stimulate further investigations of macroscopic equations that govern the fluidized state of granules through evaluations of appropriate physical properties like α and D in this study.

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APPENDIX

Calculated data for the systems (i) – (vi) reported in this manuscript are summarized in Table I. State of the system (12th column) is denoted by ○: stable convection rolls appear, △: convection rolls appear but not stable, ×: no stable convection roll exists (either partially fluidized or highly agitated).

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the conservation of energy holds (see Appendix I of [14]).

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